Lattice and q-difference Darboux-Zakharov-Manakov systems via delta -dressing method

1995 J. Phys. A: Math. Gen. 28 L173

(http://iopscience.iop.org/0305-4470/28/5/005)

J. Phys. A: Math. Gen. 28 (1995) L173–L178. Printed in the UK

LETTER TO THE EDITOR

Lattice and q-difference Darboux–Zakharov–Manakov systems via $\bar{\partial}$-dressing method

L V Bogdanov† and B G Konopelchenko

Consortium EINSTEIN‡, Dipartimento di Fisica dell'Università and Sezione INFN, 73100 Lecce, Italy

Received 12 December 1994

Abstract. A general scheme is proposed for the introduction of lattice and q-difference variables to integrable hierarchies in the frame of the $\bar{\partial}$-dressing method. Using this scheme, lattice and q-difference Darboux–Zakharov–Manakov systems of equations are derived. Darboux, Bäcklund and Combescure transformations and exact solutions for these systems are studied.

In this letter we will discuss the lattice and q-difference integrable versions of the well known Darboux–Zakharov–Manakov (DZM) system

$$\partial_i \partial_j H_k = (\partial_j H_i) H_i^{-1} \partial_i H_k + (\partial_i H_j) H_j^{-1} \partial_j H_k \tag{1}$$

where $i, j, k = 1, 2, 3$ and $i \neq j \neq k \neq i$. This system (1) is about 100 years old. It was discovered by Darboux within his study of the triply conjugate systems of surfaces [1] and since then has been studied intensively by geometrists (see e.g. [2–4]).

The system (1) was rediscovered 10 years ago by Zakharov and Manakov within the framework of a dressing method [5]. But it was only a few years ago that the inter-relation between the new results of Zakharov and Manakov and the old geometrical constructions of Darboux [6] was revealed. During the last few years the DZM system has been studied in detail: wide classes of exact solutions have been constructed [7], different transformations and reductions have been analysed [8, 9].

It should be noted that the applications of the DZM system have not been exhausted by differential geometry alone. It was discovered recently that the system (1) plays a key role in the theory of Hamiltonian and semi-Hamiltonian systems of hydrodynamical type [10–12] and in two-dimensional topological field theories [13]. The DZM system (1) also arises as the universal equations for certain hierarchies of integrable equations [14, 15].

Our study of the difference and q-difference versions of the DZM system is motivated by the well established understanding that the difference versions of integrable systems reveal the fundamental nature and algebraic structure of the corresponding continuous nonlinear integrable PDEs.

In this letter we construct the difference and q-difference DZM systems using the $\bar{\partial}$-dressing method [5], [15] (see also [7]). We discuss the first-order form of the difference and q-difference DZM system and its properties. Darboux, Bäcklund and Combescure transformations for the DZM system are derived via the $\bar{\partial}$-dressing. Exact solutions are found.

† Permanent address: IINS, Landau Institute for Theoretical Physics, Kosygin Street 2, Moscow 117940, GSP1, Russia; e-mail Leonid@landau.ac.ru

‡ European Institute for Nonlinear Studies via Transnationally Extended Interchanges.

We also derive the analogue of the Hirota bilinear identity for the difference and q-difference DZM systems (see [17]) via the $\bar{\partial}$-dressing method. Such bilinear identities can be used as the starting point of the Hirota–Sato approach to the DZM hierarchies.

The scheme of the $\bar{\partial}$-dressing method uses the non-local $\bar{\partial}$-problem with the special dependence of the kernel on additional variables

$$\begin{aligned}&\bar{\partial}(\chi(x,\lambda)-\eta(x,\lambda))=\iint_{\mathbb{C}}\mathrm{d}\mu\wedge\mathrm{d}\bar{\mu}\,\chi(\mu)g^{-1}(\mu)R(\mu,\lambda)g(\lambda))\\&(\chi(x,\lambda)-\eta(x,\lambda))_{|\lambda|\to\infty}\to 0\end{aligned} \tag{2}$$

where $\lambda\in\mathbb{C}$, $\bar{\partial}=\partial/\partial\bar{\lambda}$, $\eta(x,\lambda)$ is a rational function of λ (normalization). In this work we treat the non-commutative case, so the function $\chi(\lambda)$ and the kernel $R(\lambda,\mu)$ are matrix-valued functions.

The dependence of the solution $\chi(\lambda)$ of problem (2) on extra variables is hidden in the function $g(\lambda)$. Usually these variables are continuous space and time variables, but it is also possible to introduce discrete (lattice) and q-difference variables into the $\bar{\partial}$-dressing formalism. We will consider the following functions $g(\lambda)$:

$$g_i^{-1}=\exp(K_ix_i)\qquad \frac{\partial}{\partial x_i}g^{-1}=K_ig^{-1} \tag{3}$$

$$g_i^{-1}=(1+l_iK_i)^{n_i}\qquad \Delta_ig^{-1}=\frac{g^{-1}(n_i+1)-g^{-1}(n_i)}{l_i}=K_ig^{-1} \tag{4}$$

$$g_i^{-1}=\mathrm{e}_q(K_iy_i)\qquad \delta_i^q g^{-1}=\frac{g^{-1}(qy_i)-g^{-1}(y_i)}{(q-1)y_i}=K_ig^{-1}. \tag{5}$$

Here $K_i(\lambda)$ are meromorphic matrix functions commuting for different values of i. Function (3) introduces a dependence on the continuous variable x_i, function (4) on discrete variable n_i and function (5) defines a dependence of $\chi(\lambda)$ on the variable y_i (we will call it a q-difference variable). To introduce a dependence on several variables (maybe of a different type), one should consider a product of corresponding functions $g(\lambda)$ (all of them commute). Equations on the right-hand side of (3)–(5) and the boundary condition $g(0)=1$ characterize the corresponding functions (and give a definition of $\mathrm{e}_q(y)$). These equations play a crucial role in the algebraic scheme of constructing integrable equations in the framework of the $\bar{\partial}$-dressing method. This scheme is based on the assumption of unique solvability of problem (2) and on the existence of special operators, which transform solutions of problem (2) into the solutions of the same problem with other normalization.

We suppose that the kernel $R(\lambda,\mu)$ is equal to zero in some open subset G of the complex plane, with respect to λ and to μ. This subset should typically include all zeros and poles of the considered class of functions $g(\lambda)$ and a neighbourhood of infinity.

In this case the solution of problem (1) normalized by η is the function

$$\chi(\lambda)=\eta(x,\lambda)+\varphi(x,\lambda)$$

where $\eta(\lambda)$ is a rational function of λ (normalization), all poles of $\eta(\lambda)$ belong to G, $\varphi(\lambda)$ decreases as $\lambda\to\infty$ and is *analytic* in G.

The solutions of problem (2) with a rational normalization form a linear space, let us denote this space W. This space depends on the corresponding extra variables (in fact it is a functional of the function g). It is easy to check that

$$W(g)=gW(1). \tag{6}$$

The $\bar{\partial}$-problem (2) implies the difference and q-difference extensions of the famous Hirota bilinear identity. Indeed, let us consider problem (2) and its formal adjoint for the function normalized by $(\lambda-\mu)^{-1}$ with different functions g (i.e. with different values of coordinates):

$$\begin{aligned} \frac{\partial}{\partial\bar{\lambda}}\chi(\lambda,\mu) &= 2\pi \mathrm{i}\delta(\lambda-\mu)+\iint_{\mathbb{C}} \mathrm{d}\nu\wedge \mathrm{d}\bar{\nu}\chi(\nu,\mu)g_1(\nu)^{-1}R(\nu,\lambda)g_1(\lambda) \\ \frac{\partial}{\partial\bar{\lambda}}\chi^*(\lambda,\mu) &= -2\pi \mathrm{i}\delta(\lambda-\mu)-\iint_{\mathbb{C}} \mathrm{d}\nu\wedge \mathrm{d}\bar{\nu}g_2(\mu)^{-1}R(\lambda,\nu)g_2(\nu)\chi^*(\nu,\mu). \end{aligned} \tag{7}$$

After simple calculations (in the case of continuous variables see [18]) we obtain

$$\int_{\gamma}\chi(\nu,\lambda;g_1)g_1^{-1}(\nu)g_2(\nu)\chi^*(\nu,\mu;g_2)\,\mathrm{d}\nu = 0 \tag{8}$$

where γ is the boundary of G. It follows from (8) that in $\bar{G}$ the function $\chi(\lambda,\mu)$ is equal to $-\chi^*(\mu,\lambda)$, so in fact this identity should be written for one function. It is possible to take identity (8) instead of (2) as the starting point for the algebraic scheme of constructing equations.

The algebraic scheme of constructing equations is based on the following property of problem (2) with the dressing functions (3)–(5): if $\chi(\boldsymbol{x},\boldsymbol{n},\boldsymbol{y},\lambda)\in W(\boldsymbol{x},\boldsymbol{n},\boldsymbol{y})$, then the functions

$$\begin{aligned} D_i^c\chi &= \partial/\partial x_i\chi+\chi K_i(\lambda) \\ D_i^d\chi &= \Delta_i\chi+T_i^d\chi K_i(\lambda) \\ D_i^q\chi &= \delta_i^q\chi+T_i^q\chi K_i(\lambda) \end{aligned} \tag{9}$$

also belong to W, where $Tf(n)=f(n+1)$, $T^q f(y)=f(qy)$. We can multiply the solution from the left by the arbitrary matrix function of additional variables, $u(\boldsymbol{x},\boldsymbol{n},\boldsymbol{y})\chi\in W$. So the operators (9) are the generators of a Zakharov–Manakov ring of operators, that transform W into itself.

Combining this property with the unique solvability of problem (1), one obtains the differential relations between the coefficients of expansion of functions $\chi(\boldsymbol{x},\boldsymbol{n},\boldsymbol{y},\lambda)$ into powers of $(\lambda-\lambda_p)$ at the poles of $K_i(\lambda)$ [15].

The derivation of equations in this case is completely analogous to the continuous case [5]. First we choose three functions $K_i(\lambda)$, $K_j(\lambda)$ and $K_k(\lambda)$ in the form

$$K_i(\lambda)=\frac{A_i}{\lambda-\lambda_i} \tag{10}$$

where A_i, A_j and A_k are commuting matrices, $\lambda_i\neq\lambda_j\neq\lambda_k\neq\lambda_i$. Then we introduce the solution of problem (2) $\chi(\lambda)$ with the canonical normalization ($\eta(\lambda)=1$). The following derivation will be conducted for the q-difference case; to get the difference case you should just change δ_i^q for Δ_i and T_i for T_i^q.

The function χ satisfies the linear equations

$$D_i^qD_j^q\chi = T_i^q((D_j^q(\lambda_i)\chi_i)\chi_i^{-1})D_i^q(\lambda_k)\chi+T_j^q((D_i^q(\lambda_j)\chi_j)\chi_j^{-1})D_j^q(\lambda_k)\chi \tag{11}$$

where $\chi_i=\chi(\lambda_i)$. Evaluating equation (11) at the point λ_k, we obtain the closed system of equations for the functions χ_i:

$$D_i^q(\lambda_k)D_j^q(\lambda_k)\chi_k = T_i^q((D_j^q(\lambda_i)\chi_i)\chi_i^{-1})D_i^q(\lambda_k)\chi_k+T_j^q((D_i^q(\lambda_j)\chi_j)\chi_j^{-1})D_j^q(\lambda_k)\chi_k. \tag{12}$$

It is possible to transform the operators D to usual derivatives or difference operators by the substitution

$$\chi_k = H_k g_i(\lambda_k)g_j(\lambda_k) \qquad \psi=\chi g_i(\lambda)g_j(\lambda)g_k(\lambda) \tag{13}$$

(this substitution works for all cases—continuous, difference and q-difference, you should only take the corresponding function (3)–(5)). Then the linear equations (11) and the equations for the functions H_i read

$$\delta_i^q \delta_j^q \psi = T_i^q((\delta_j^q H_i)H_i^{-1})\delta_i^q \psi + T_j^q((\delta_i^q H_j)H_j^{-1})\delta_j^q \psi \tag{14}$$

$$\delta_i^q \delta_j^q H_k = T_i^q((\delta_j^q H_i)H_i^{-1})\delta_i^q H_k + T_j^q((\delta_i^q H_j)H_j^{-1})\delta_j^q H_k. \tag{15}$$

Equations (15) represent a q-difference integrable deformation of a Zakharov–Manakov system.

System (15) can be rewritten in first-order form by the substitution

$$\beta_{ij} = (T_i^q H_i)^{-1}\delta_i^q H_j. \tag{16}$$

Using the identity $\delta^q(g^{-1}) = -(T^q g)^{-1}(\delta^q g)g^{-1}$ we obtain the equation

$$\delta_k^q \beta_{ij} = (T_k^q \beta_{ik})\beta_{kj}. \tag{17}$$

Correspondingly the linear system (14) becomes

$$\delta_k^q \psi_i = (T_k^q \beta_{ik})\psi_k \tag{18}$$

where $\psi_i = (T_i^q H_i)^{-1}\delta_i^q \psi$. Note that the equations (17) and (18) can be derived directly from the bilinear identity (8). In this case the subset G consists of the neighbourhoods of the points $\lambda_i, \lambda_j, \lambda_k$ and $\lambda = \infty$.

The continuous version of the system (17) has important applications in the theory of systems of hydrodynamical type and topological field theory [13]. We hope that similar applications will be found for equations (17) too.

Now we will demonstrate how certain symmetry transformations for the DZM system can be derived via the $\bar\partial$-dressing method.

Let us introduce the function $\tilde g(\lambda)$ in addition to the functions g_i, g_j, g_k,

$$\tilde g(\lambda) = \frac{\lambda - \lambda_i}{\lambda - \tilde\lambda} \tag{19}$$

where $\tilde\lambda \in G$. It follows from (6) that

$$\tilde W(y_i, y_j, y_k) = \tilde g W(y_i, y_j, y_k). \tag{20}$$

Let the canonically normalized function χ in the space W be given; it satisfies linear equations (11). Property (20) gives an opportunity to calculate the canonically normalized function $\tilde\chi$ in the space $\tilde W$ in terms of χ, this function also satisfies equations (11) (with other potentials). It implies that

$$\tilde\chi = u(y)\tilde g\chi + v(y)\tilde g D_i^q \chi \tag{21}$$

with the properly chosen functions u and v. Using two conditions: the absence of poles and unit asymptotics at infinity, we get

$$\tilde\chi = \tilde g(\chi - \chi(\tilde\lambda)(D_i^q\chi)^{-1}(\tilde\lambda)D_i^q\chi). \tag{22}$$

If one transforms operators D_i^q into q-difference operators δ_i^q by the substitution $\chi = \psi g_i g_j g_k$, one obtains

$$\tilde\psi(\lambda) = \tilde g(1 - \psi(\tilde\lambda)(\delta_i^q\psi)^{-1}(\tilde\lambda)\delta_i^q)\psi(\lambda) \tag{23}$$

which is the well known Darboux transformation for the DZM system (see [1, 8 and 9]).

Introduction of group element (19) may be treated in a different manner, namely as the introduction of an extra discrete variable with

$$\tilde D = \tilde\Delta + \tilde K\tilde T \qquad \tilde K = \frac{\lambda_i - \tilde\lambda}{\lambda - \lambda_i}. \tag{24}$$

In this notation $\tilde{\chi} = \tilde{T}\chi$. Using the formulae (15) for two q-difference variables y_i, y_j and discrete variable $\tilde{n}$, we obtain the following transformation for the q-deformation of DZM system (15)

$$\tilde{\Delta}\delta_j^q H_i = \tilde{T}((\delta_j^q H_k)H_k^{-1})\tilde{\Delta}H_i + T_j^q((\tilde{\Delta}H_j)H_j^{-1})\delta_j^q H_i \tag{25}$$

$$\delta_i^q \tilde{\Delta}H_j = T_i^q((\tilde{\Delta}H_i)H_i^{-1})\delta_i^q H_j + \tilde{T}((\delta_i^q H_k)H_k^{-1})\tilde{\Delta}H_j. \tag{26}$$

This transformation establishes a connection between two solutions of the system (15): H_i, H_j, $H_k \to \tilde{T}H_i$, $\tilde{T}H_j$, $\tilde{T}H_k$ and it is nothing but the Bäcklund transformation.

To derive the Combescure transformation in the framework of the $\bar{\partial}$-dressing method, it is necessary to use the freedom to choose a normalization of problem (2) in a quite non-trivial way. In this section we consider the commutative case of the $\bar{\partial}$-dressing method, so all functions take their values in $\mathbb{C}$. Let us introduce the solution of problem (2) $\chi(\lambda, \mu)$ normalized by $(\lambda - \mu)^{-1}$, where μ is a parameter, $\mu \in G$, and let us modify operators D by just adding constants $c_i = \frac{A_i}{(\lambda_i - \mu)}$ to them

$$D_i'^q = D_i^q + c_i = D_i^q + \frac{A_i}{(\lambda_i - \mu)} = \delta_i^q - \frac{A_i}{(\lambda_i - \mu)}(T_i^q - 1) + A_i \frac{\lambda - \mu}{(\lambda - \lambda_i)(\lambda_i - \mu)} T_i^q. \tag{27}$$

We would like to emphasize that the kernel of problem (2) remains the same. Then the function $\chi(\lambda, \mu)$ satisfies equation (12) with the modified operators D'. To transform operators D' to q-difference operators in this case one should use a substitution

$$\begin{aligned} \chi_k &= H_k g_i(\lambda_k) e_q^{-1}(c_i y_i) g_j(\lambda_k) e_q^{-1}(c_j y_j), \\ \psi &= \chi g_i(\lambda) g_j(\lambda) g_k(\lambda) e_q^{-1}(c_i y_i) e_q^{-1}(c_j y_j) e_q^{-1}(c_k y_k). \end{aligned} \tag{28}$$

So by using different normalizations we obtain different solutions of the system (15). It happens that the connection between these solutions is given by the Combescure transformation. Indeed, unique solvability of problem (2) implies that $D_i'^q \chi(\lambda, \mu) = u_i(y) D_i^q \chi(\lambda)$ or, after substitution (28),

$$\delta_i^q \psi(\lambda, \mu) = U_i(y) \delta_i^q \psi(\lambda). \tag{29}$$

The compatibility conditions for equations (29) give the q-deformation of equations of Combescure transformation [1]

$$\delta_j^q U_i = (T_i U_j - T_j U_i) T_i^q ((\delta_j^q H_i) H_i^{-1}). \tag{30}$$

In the continuous case these equations are important for the connection with the systems of hydrodynamical type [10–12].

The procedure for obtaining exact solutions in the framework of the $\bar{\partial}$-dressing method is based on the fact that for the degenerate kernel $R(\lambda, \mu)$, problem (2) is explicitly solvable. This property also holds for the case of lattice and q-difference variables. Let us apply it to a simple example.

There is one important special case of the non-local $\bar{\partial}$-problem which is exactly solvable, which corresponds to plane soliton solutions. This is the case of δ-functional kernels

$$R(\lambda, \mu) = 2\pi \mathrm{i} \sum_{\alpha=1}^{N} R_\alpha \delta(\lambda - \lambda_\alpha) \delta(\mu - \mu_\alpha) \tag{31}$$

where λ_α, μ_α are a set of points in the complex plane, $\lambda_\alpha \neq \mu_{\alpha'}$,

$$R_\alpha = (g_i g_j g_k)^{-1}(\mu_\alpha) C_\alpha g_i g_j g_k(\lambda_\alpha).$$

In this case the solution of problem (1) is a rational function, and problem (1) reduces to the system of linear equations. As a result the canonically normalized function χ is given by

$$\chi(\lambda) = 1 - \sum_{\alpha,\alpha'} ((A')^{-1})_{\alpha\alpha'} \frac{1}{(\lambda - \lambda_\alpha)} \tag{32}$$

$$A'_{\alpha\alpha'} = R_\alpha^{-1} \delta_{\alpha\alpha'} - \frac{1}{\mu_\alpha - \lambda_{\alpha'}}.$$

To get solutions of the DZM equations (15) from this formula, one should use relation $\chi_i = \chi(\lambda_i)$, substitution (13) and corresponding explicit expressions for the functions g_i.

This work was supported in part by INTAS (an international association for the promotion of cooperation with scientists from independent states of the former Soviet Union). The first author (LB) also acknowledges partial support by the Soros Foundation (ISF) (grant MLY000) and the Russian Foundation for Fundamental Studies (grant 94-01-00899).

References

[1] Darboux G 1910 *Lecons sur les Systémes Orthogonaux et les Coordonneés Curvilignes* (Paris)
[2] Bianchi L 1902 *Lezioni di Geometria Differentiale* 2nd edn (Piza: Spoerri)
[3] Eisenhart L P 1909 *A Treatise on the Differential Geometry of Curves and Surfaces* (New York: Dover)
[4] Forsyth A R 1920 *Lectures on the Differential Geometry of Curves and Surfaces* (Cambridge: Cambridge University Press)
[5] Zakharov V E and Manakov S E 1984 *Zap. N. S. LOMI* **133** 77; 1985 *Funk. Anal. Ego Prilozh.* **19** 11
[6] Dryuma V S 1991 *Proc. NEEDS Workshop (Dubna 1990)* (Singapore: World Scientific) p 94; 1992 *Math. Studies* vol 124 (Moscow: Kishinev) p 56
[7] Konopelchenko B G 1993 *Solitons in Multidimensions* (Singapore: World Scientific)
[8] Konopelchenko B G and Schief W K 1993 Lamé and Zakharov–Manakov systems: Combescure, Darboux and Bäklund transformations *Preprint* AM 93/9, UNSW, Sydney
[9] Schief W K 1994 *Inverse Problems* **10** 1185
[10] Tsarev S P 1991 *Math. USSR Izv.* 37(2) 347
[11] Dubrovin B A and Novikov S P 1989 *Usp. Mat. Nauk* **44** 29
[12] Tsarev S P 1992 Classical differential geometry and integrability of systems of hydrodynamical type (Exeter workshop 1992) *Preprint* hep-th 9303092
[13] Dubrovin B 1992 *Nucl. Phys.* B **379** 627; 1993 *Commun. Math. Phys.* **152** 539; 1994 Geometry of 2D topological field theories *Preprint* SISSA-89/94/FM
[14] Nijhoff F and Maillet J-M 1988 Nonlinear evolution *Proc. IV NEEDS (1987)* ed J J P Léon (Singapore: World Scientific) p 281
[15] Bogdanov L V and Manakov S V 1988 *J. Phys. A: Math. Gen.* **21** 537
[16] Zakharov V E 1990 On the dressing method *Inverse Methods in Action* ed P C Sabatier (Berlin: Springer)
[17] Bogdanov L V 1994 *Teor. Mat. Fiz.* **99** 177; 1994 talk at NLS conference (Chernogolovka) to be published
[18] Carroll R and Konopelchenko B 1993 *Lett. Math. Phys.* **28** 307